RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College under University of Calcutta)

M.A./M.SC. THIRD SEMESTER EXAMINATION, DECEMBER 2012

SECOND YEAR

MATHEMATICS

Date : 13/12/2012

Time : 11 am – 2 pm

Paper : X

Full Marks : 60

(Use separate answer-book for each group)

Group – A

(Classical Mechanics)

(Answer **Question No. 1** and **any three** from the rest)

1. Attempt **any one**:

   a) A particle is projected vertically upwards from a point O on the earth's surface. Taking the rotation of the earth about its axis into account, show that when the particle falls to the ground, it is deviated towards the west through an amount

   $$\frac{4}{3}\frac{V^3}{g^2}\omega\cos\lambda$$

   where V is the upward velocity of projection, ω is the angular velocity of the earth about its axis and λ is the geographical latitude of the place. [6]

   b) If forces are conservative and geometrical equations do not contain time t explicitly, then by help of Lagrange's equations of motion prove that $T + V =$ constant where $T = K.E.$ and $V = P.E.$ of the system. [6]

2. A smooth wire bent in the form of a parabola has for its equation $z = ay^2$ when the wire lies in the yz-plane with z-axis vertical. A bead slides on the wire and the wire rotates with constant angular velocity ω about the vertical z-axis. Deduce the Hamiltonian (H) and obtain the equation of motion. Is $H = E$, where E is the total energy? [8]

3. A bead of mass m slides on a smooth uniform circular wire of radius $r = a$ which is rotating with constant angular velocity ω about the diameter which is vertical. Set up the Lagrangian and find the equation of motion of the bead.

   If the bead is released from rest from a point on the level of the centre of the circular wire, show that it will not reach the lowest point if $\omega > \sqrt{\frac{2g}{a}}$. [8]

4. In a dynamical system with two degrees of freedom, the kinetic energy T is given by

   $$T = \frac{\dot{q}_1^2}{2(a+bq_2)} + \frac{1}{2}q_2^2\dot{q}_2^2$$

   and the potential energy $V = c + dq_2$ where q_1, q_2 are generalized coordinates and a, b, c, d are constants. Use the method of ignoration of coordinates to show that the value of q_2 in terms of time t is given by equation of the form

   $$(q_2 - k)(q_2 + 2k)^2 = h_1(t - t_0)^2$$

   where h_1, k, t_0 are constants. [8]

5. State Hamilton's Principle. Derive Hamilton's canonical equations of motion using Hamilton's Principle. [8]

6. O and O' are two fixed points in space not in the same vertical line, O being at higher level than O'. Find the form of the plane curve joining O and O' such that the time of fall of a smooth particle from O to O' along the curve starting from rest at O and moving under gravity is minimum. [8]

Group – B

(PDE)

[*The symbols have their usual meanings*]

Section - I

(Answer **any three** questions) 3x5

7. Establish the recurrence relation $xP_n'(x) - P_{n-1}'(x) = nP_n(x)$.

8. Show that $\int_{-1}^{1} x^m P_n(x)\,dx = \begin{cases} 0 & , m < n \\ \dfrac{2^n (\underline{|n})^2}{\underline{|2n}} \cdot \dfrac{2}{2n+1}, & m = n \end{cases}$.

9. When n is an integer, show that $J_n(x)$ is equal to the coefficient of t^n in the expansion of $e^{x/2\left(t-\frac{1}{t}\right)}$.

10. Evaluate $\int_0^x x^3 J_0(x)\,dx$.

11. If $a > 0$, show that $\int_0^\infty e^{-ax} J_0(bx)\,dx = \dfrac{1}{\sqrt{a^2+b^2}}$.

Section - II

(Answer **any three** questions)

12. Reduce the equation $(n-1)^2 u_{xx} - y^{2n} u_{yy} = n y^{2n-1} u_y$ when $n\ (\neq 1)$ is an integer, to canonical form and find its general solution. [5]

13. For the linear PDE $Lu \equiv \sum_{i,j=1}^{n} a_{ij} \dfrac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \dfrac{\partial u}{\partial x_i} + cu = 0$ where $a_{ij} (= a_{ji}), b_i, c$ are functions of $x_1, x_2, \cdots, x_n$ having continuous second order partial derivatives in $D \subset E^n$; prove that L will be self-adjoint if $\sum_{j=1}^{n} \dfrac{\partial a_{ij}}{\partial x_j} = b_i\ \forall\, i = 1(1)n$. [5]

14. Solve the PDE, $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0$ where u is finite at $r = 0$, $u(a,\theta) = U_0 (=\text{constant})$ and $u(r,0) = u(r,\pi) = 0$ by the method of separation of variables. [5]

15. Using $u(x,t) = f(x+ct) + g(x-ct)$ as the solution of $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$, find D'Alembert's solution when $u(x,0) = u_0(x), \dfrac{\partial u}{\partial t}(x,0) = u_1(x)$. Also show that this solution is unique. [5]

16. Find the general integral of the equation $(x-y)u_x + (y-x-u)u_y = u$, given that $u = 1$ on $\Gamma : x^2 + y^2 = 1$. [5]